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Exact solutions for nonlinear diffusion with first-order loss

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Abstract—Exact solutions are developed for nonlinear diffusion with first-order loss (e.g. by reaction, irreversible absorption, biological degradation, or radioactive decay) from an instantaneous source. The diffusivity is proportional to a positive power of concentration. The solutions are for an arbitrary number of dimensions $s > 0$, with $s = 1, 2, 3$ in physical applications. All solutions give the slug radius exponentially approaching a finite maximum, with concentration decreasing exponentially to zero. Applications include the gravity spreading, and ultimate extinction, of liquid lenses on solid, or immiscible liquid, surfaces. The corresponding exact solutions for first-order gain, not loss, are also given.

1. INTRODUCTION

FOR DIFFUSION with diffusivity a positive power-law of concentration, an instantaneous source spreads to a slug which is always of finite radius at finite time [1–4]. If material is lost from the slug (e.g. by chemical reaction, absorption on a porous substrate, solution, or evaporation), it may shrink and vanish in finite time. It may, on the other hand, approach a finite radius, with concentration vanishing, in the limit of infinite time [5].

We pursue the matter here through study of the equation

$$\frac{\partial\theta}{\partial t_*} = r_*^{1-s}\frac{\partial}{\partial r_*}\left(r_*^{s-1}D_1\theta^m\frac{\partial\theta}{\partial r_*}\right) - k\theta. \qquad (1)$$

Here θ ($\geqslant 0$) is concentration, normalized with respect to some standard concentration, t_* is time, r_* is the radial space coordinate ($0 \leqslant r_* \leqslant \infty$), and s is the number of space dimensions 1, 2, or 3. (The analysis holds, in fact, for all $s > 0$.) The diffusivity takes the power-law form $D_1\theta^m (D_1 > 0, m \geqslant 0)$; and the time-rate of material loss, $k\theta$ ($k > 0$) describes first-order loss by, for example, chemical reaction, irreversible absorption, biological degradation, or radioactive decay. The substitutions

$$r = \left(\frac{k}{D_1}\right)^{1/2} r_*, \quad t = kt_* \qquad (2)$$

yield the dimensionless form

$$\frac{\partial\theta}{\partial t} = r^{1-s}\frac{\partial}{\partial r}\left(r^{s-1}\theta^m\frac{\partial\theta}{\partial r}\right) - \theta. \qquad (3)$$

We seek solutions of (3) satisfying the initial condition

$$t = 0, \quad 0 \leqslant r \leqslant \infty, \quad \theta = Q\delta(r), \qquad (4)$$

with Q the dimensionless source strength ($0 < Q < \infty$), and with $\delta(\ )$ defined by a limiting process to be determined. The physical instantaneous source strength with dimensions [length]^s, $Q_* = (D_1/k)^{s/2}Q$.

The solution for the linear case with $m = 0$, found by well-known methods [6], is

$$\theta = \frac{Q}{(4\pi t)^{s/2}}\exp\left[-\frac{r^2+4t^2}{4t}\right]. \qquad (5)$$

For gain in place of loss (Section 5) $+4t^2$ in (5) is replaced by $-4t^2$. In what follows we consider the linear case no further; and all results are for $m > 0$.

2. SOLUTIONS OF EQUATIONS (3) AND (4)

We observe that the substitutions

$$\theta(r,t) = u(r,\tau)\,e^{-t}, \qquad (6)$$

$$\tau = m^{-1}(1-e^{-mt}), \quad 0 \leqslant \tau \leqslant m^{-1}, \qquad (7)$$

give

$$\frac{\partial\theta}{\partial t} = \frac{d\tau}{dt}\frac{\partial u}{\partial\tau}e^{-t} - u\,e^{-t} = e^{-(m+1)t}\frac{\partial u}{\partial\tau} - \theta \qquad (8)$$

and

$$r^{1-s}\frac{\partial}{\partial r}\left(r^{s-1}\theta^m\frac{\partial\theta}{\partial r}\right) = e^{-(m+1)t}r^{1-s}\frac{\partial}{\partial r}\left(r^{s-1}u^m\frac{\partial u}{\partial r}\right). \qquad (9)$$

Putting (8) and (9) in (3) reduces it to

$$\frac{\partial u}{\partial\tau} = r^{1-s}\frac{\partial}{\partial r}\left(r^{s-1}u^m\frac{\partial u}{\partial r}\right). \qquad (10)$$

Since (6) and (7) imply that at $t = 0$, $\tau = 0$ and $u = \theta$, (10) is subject to the initial condition

$$\tau = 0, \quad 0 \leqslant r \leqslant \infty, \quad u = Q\delta(r). \qquad (11)$$

NOMENCLATURE

D_1	diffusivity for $\theta = 1$ [$m^2 s^{-1}$]
g	gravitational acceleration [$m s^{-2}$]
k	coefficient of rate of material loss (or gain) [s^{-1}]
m	power index of diffusivity
q	dimensionless quantity of material in slug
Q	dimensionless instantaneous source strength
Q_*	physical instantaneous source strength [m^s]
r	dimensionless radial coordinate
r_0	dimensionless slug radius
r_{max}	maximum dimensionless slug radius
r_*	physical radial coordinate [m]
R	variable proportional to slug radius, defined in (17)
R_{max}	maximum R-value
s	number of space dimensionless (1, 2, or 3 in physical applications)
t	dimensionless time
$t_{\theta max}$	t-value when θ reaches its maximum at a fixed r
t_*	physical time [s]
T	variable proportional to slug concentration, defined in (16)
T_{min}	minimum T-value during diffusion with gain
u	transformed concentration-like variable, defined in (6) or (41)
U	similarity variable, defined in (13)
U_0	U-value for $\rho = 0$.

Greek symbols

γ_0	density of liquid lens [$kg m^{-3}$]
γ_1	density of underlying liquid [$kg m^{-3}$]
Γ	gamma function
δ	delta function
θ	normalized concentration (normalized lens thickness in (4))
θ_{max}	maximum θ-value at a fixed r
λ	normalization length for θ in (4) [m]
ν	kinematic viscosity of lens liquid [$m^2 s^{-1}$]
ρ	similarity variable, defined in (14)
ρ_0	ρ-value where U becomes zero
τ	transformed time-like variable, defined in (7) or (42).

The solution of (10), (11) is well known [1–4]. It is

$$0 \leqslant \rho \leqslant \left[\frac{2(sm+2)U_0^m}{m}\right]^{1/2} = \rho_0,$$

$$U = U_0\left[1-\left(\frac{\rho}{\rho_0}\right)^2\right]^{1/m};$$

$$\rho > \rho_0, \quad U = 0. \tag{12}$$

We use here the similarity substitutions

$$u = U(\rho)\tau^{-s/(sm+2)}; \tag{13}$$

$$\rho = r\tau^{-1/(sm+2)}. \tag{14}$$

U_0 is the value of U at $\rho = 0$, and its relation to Q is

$$Q = \frac{\Gamma(m^{-1}+1)}{\Gamma(\frac{1}{2}s+m^{-1}+1)}\left[\frac{2\pi(sm+2)}{m}\right]^{s/2} U_0^{(sm+2)/2}. \tag{15}$$

Miller and van Duijn [5] previously developed the solution for the special case $(s, m) = (2, 1)$.

Using (6) and (7), we express substitutions (13) and (14) in terms of θ and t:

$$\theta = U(\rho)T(t), \quad T(t) = e^{-t}[m^{-1}(1-e^{-mt})]^{-s/(sm+2)}; \tag{16}$$

$$\rho = r/R(t), \quad R(t) = [m^{-1}(1-e^{-mt})]^{1/(sm+2)}. \tag{17}$$

It follows from (16), (17) that the dimensionless quantity of material in the slug at time t,

$$q(t) = Qe^{-t}. \tag{18}$$

This is as it should be, since material is lost through a first-order process.

3. PHYSICAL IMPLICATIONS OF THE SOLUTIONS

Various implications follow.

3.1. *Time-course of central concentration*

The maximum concentration is at the slug center, $r = 0$. It follows from (16) that

$$\theta(0, t) = U_0 T(t). \tag{19}$$

In the limit of small t

$$T(t) \approx t^{-s/(sm+2)}\left(1-\frac{(sm+4)t}{2(sm+2)}\right). \tag{20}$$

At small enough t this approximates classical behavior ($k = 0$), with $T = t^{-s/(sm+2)}$ for all $t \geqslant 0$ [1–4]. Initially material loss has negligible effect. In the limit of large t,

$$T(t) \approx m^{s/(sm+2)}e^{-t}\left[1+\frac{s}{sm+2}e^{-mt}\right]. \tag{21}$$

The power-law decrease seen at small t has given way to exponential decrease dominated by exponential material loss. See Fig. 1.

3.2. *Time-course of slug radius*

It follows from (12) and (17) that the slug radius

$$r_0(t) = \rho_0 R(t). \tag{22}$$

In the limit of small t

$$R(t) \approx t^{1/(sm+2)}\left(1-\frac{m}{2(sm+2)}t\right), \tag{23}$$

which should be compared with the classical behavior ($k = 0$) with $R = t^{1/(sm+2)}$ [1–4]. Here also material loss has negligible effect initially.

In the limit of large t

$$R(t) \approx m^{-1/(sm+2)}\left[1-\frac{e^{-mt}}{sm+2}\right]. \tag{24}$$

The power-law increase has disappeared, and is supplanted by exponential approach to a finite maximum

$$R_{\max} = \lim_{t\to\infty} R(t) = m^{-1/(sm+2)}. \tag{25}$$

The maximum slug radius $r_{\max} = \rho_0 R_{\max}$. See Fig. 2.

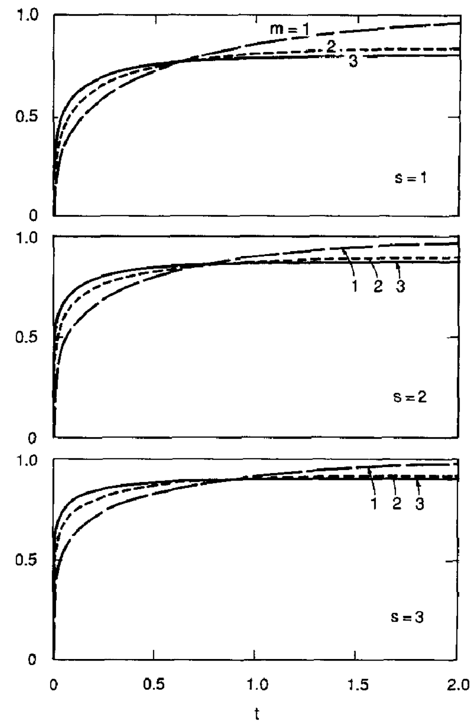

FIG. 2. The function $R(t)$ for $s = 1, 2, 3$, $m = 1, 2, 3$. The slug length-scale is proportional to R.

3.3. *Concentration envelope*

For all $0 < r < r_{\max}$, the concentration at fixed r increases from zero at $t = 0$, passes through a maximum, and returns to zero as $t \to \infty$. The maximum value of θ at fixed r, $\theta_{\max}(r)$, is important in some contexts; and the dimensionless time at which this maximum occurs, $t_{\theta\max}(r)$, is also of interest. The curve $\theta_{\max}(r)$ is the envelope of all concentration profiles in $0 \leqslant t \leqslant \infty$.

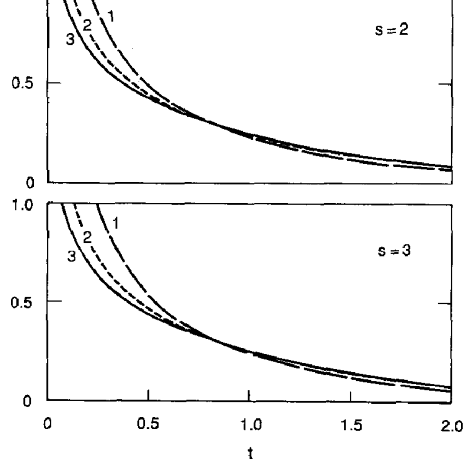

FIG. 1. The function $T(t)$ for $s = 1, 2, 3$, $m = 1, 2, 3$. The concentration scale is proportional to T, and t is dimensionless time.

Combining (12), (16), (17), we find

$$\theta(r,t) = U_0\,e^{-t}\left[\,[m^{-1}(1-e^{-m})]^{-sm/(sm+2)} - \frac{mr^2}{2(sm+2)U_0^m}[m^{-1}(1-e^{-mt})]^{-1}\right]^{1/m}. \tag{26}$$

Differentiating (26), and putting $\partial\theta/\partial t = 0$, we obtain the relation between $t_{\theta\max}$ and r,

$$r = \left\{\frac{2(sm+2)U_0^m}{m}\left[1-\frac{2}{sm+2}e^{-mt_{\theta\max}}\right]\right\}^{1/2} \times [m^{-1}(1-e^{-mt_{\theta\max}})]^{1/(sm+2)} \tag{27}$$

$$= r_{\max}\left[1-\frac{2}{sm+2}e^{-mt_{\theta\max}}\right]^{1/2}[1-e^{-mt_{\theta\max}}]^{1/(sm+2)}. \tag{28}$$

This gives $t_{\theta\max}(r)$ in inverse form. Putting the value of r from (27) in (26), we find

$$\theta[r(t_{\theta\max}), t_{\theta\max}] = \left(\frac{2}{sm+2}\right)^{1/m} U_0\, e^{-2t_{\theta\max}} \times [m^{-1}(1-e^{-mt_{\theta\max}})]^{s/(sm+2)}. \tag{29}$$

Finally, we obtain $\theta_{max}(r)$ by eliminating $t_{\theta max}$ between (27) and (29).

We observe that, in the limit as t and $t_{\theta max}$ approach zero, (26), (27), (29) become

$$\theta(r,t) = U_0 t^{-A}\left[1-\frac{mr^2}{(2sm+4)U_0^m t^{2/(sm+2)}}\right]^{1/m},$$

$$r = [2sU_0^m]^{1/2} t_{\theta max}^{1/(sm+2)},$$

$$\theta[r(t_{\theta max}), t_{\theta max}] = \left[\frac{2}{sm+2}\right]^{1/m} U_0 t_{\theta max}^{-A}; \quad (30)$$

and that

$$\theta_{max}(r) = (2s)^{s/2}\left[\frac{2}{sm+2}\right]^{1/m} U_0^{(sm+2)/2} r^{-s}. \quad (31)$$

In view of (15), (31) implies $\theta_{max}(r) \propto Qr^{-s}$. Note that (30) and (31) represent, as they should, the exact results for all $t \geqslant 0$ for the classical material-conserving ($k = 0$) solutions [1–4]. See Fig. 3.

We observe further that, in the limit as $r \to r_{max}$,

$$\theta_{max}(r) = m^{s/(sm+2)}\left(\frac{sm+2}{2}\right)^{1/m} U_0 \left(\frac{r_{max}-r}{r_{max}}\right)^{2/m}. \quad (32)$$

3.4. *Illustrative results*

Figure 1 shows graphs of $T(t)$ for $s = 1, 2, 3$ and $m = 1, 2, 3$. As we have seen, $T(t)$ represents in reduced form the evolution of the concentration scale within the slug.

Similarly, in Fig. 2 we depict the evolution of slug radius (or length scale) in the normalized form $R(t)/R_{max}$. The figure shows results for $s = 1, 2, 3$ and $m = 1, 2, 3$.

Figure 3 shows an example of the concentration envelope, calculated from (27) and (29). It is presented in the reduced form θ_{max}/U_0 as a function of r/r_{max}. The curve is for $(s, m) = (2, 2)$. The envelope for material conservation ($k = 0$) is shown for comparison. The envelope for diffusion with loss deviates significantly from that for $k = 0$ when r exceeds about 0.4 r_{max}.

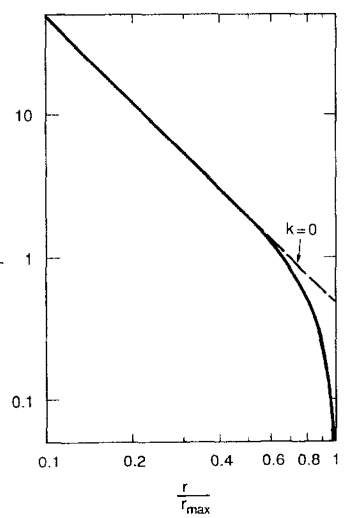

FIG. 3. The concentration envelope for $(s, m) = (2, 2)$ in the reduced form θ_{max}/U_0 as a function of r/r_{max}. The broken line is the envelope for material conservation ($k = 0$).

4. APPLICATION TO GRAVITY SPREADING OF LIQUID LENSES

4.1. *Gravity spreading on solid and immiscible liquid surfaces*

In 1956 Philip [7] showed that the viscous flow of a liquid film over a horizontal solid surface may be described as nonlinear diffusion. The normalized film thickness θ is the concentration-like variable, and the diffusivity is $g\lambda^3\theta^3/(3\nu)$. Here g is the gravitational acceleration, ν the kinematic viscosity, and λ the normalization length. The analysis assumes that inertial effects are trivially small, that flow velocities are horizontal, and that effects due to vapor transport, surface tension, and surface diffusion are negligible.

In 1976 Lopez *et al.* [8] applied this formulation to gravity spreading of a liquid lens on such a surface, in the absence of reaction or absorption, see Fig. 4(a). (The assumption of horizontal flow velocities is not satisfied at small times after release of the lens. Strictly, the 'negligible inertia' assumption fails then also, but this limitation is less serious. Evidently the diffusion formulation holds for this application least well in the very early stages of the spreading process.)

The same analysis applies to gravity spreading of a liquid lens (density γ_0) on the surface of an underlying immiscible liquid (density $\gamma_1 > \gamma_0$). Spreading of oil on water is an example. The depth of the underlying liquid is taken to be large and the pressure in it hydrostatic, unaffected by the lens at its surface, see Fig. 4(b). Here also the normalized lens depth is θ, but now its upper surface is at normalized elevation $(1-\gamma_0/\gamma_1)\theta$ above the plane horizontal surface of the second liquid which lies outside the lens; and its lower

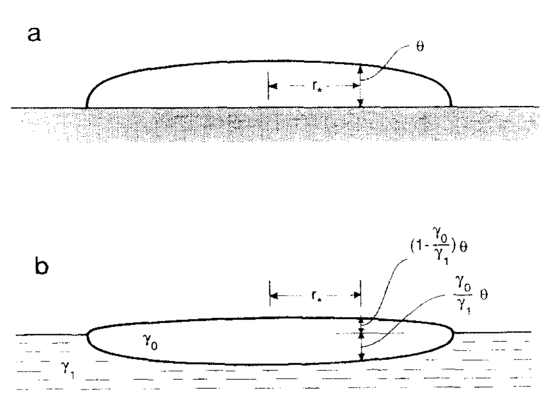

FIG. 4. (a) Liquid lens on solid surface. (b) Liquid lens on immiscible liquid surface. The lens density γ_0 is less than the density of the underlying liquid, γ_1.

surface is at normalized elevation $\gamma_0\theta/\gamma_1$ below that plane. These relations are analogous to those of the Ghyben–Herzberg lens [9, 10] of fresh/salt groundwater relations in hydrology. For this case, under the foregoing assumptions, the flow is described by the same formulation, except that now the diffusivity is $(1-\gamma_0/\gamma_1)g\lambda^3\theta^3/(3\nu)$.

The analysis is readily extended to embrace first-order material loss. The relevant equation is thus (1), with $m = 3$ and $D_1 = g\lambda^3/(3\nu)$ for lenses on solid surfaces and $(1-\gamma_0/\gamma_1)\, g\lambda^3/(3\nu)$ for lenses on liquid surfaces, subject to initial condition (4). Two of the foregoing solutions (both represented in Figs. 1 and 2) are relevant.

4.2. *Axisymmetric spreading*, $(s, m) = (2, 3)$

For axisymmetric gravity spreading of the lens, with first-order loss, $(s, m) = (2, 3)$, yielding:

$$0 \leqslant \rho \leqslant \left[\frac{16}{3} U_0^3\right]^{1/2} = \rho_0,$$

$$U(\rho) = U_0\left[1-\left(\frac{\rho}{\rho_0}\right)^2\right]^{1/3}. \tag{33}$$

$$Q = 4\pi U_0^4. \tag{34}$$

$$T(t) = \mathrm{e}^{-t}[(1-\mathrm{e}^{-3t})/3]^{-1/4} \tag{35}$$

$$R(t) = [(1-\mathrm{e}^{-3t})/3]^{1/8}. \tag{36}$$

4.3. *Linear spreading*, $(s, m) = (1, 3)$

For one-dimensional gravity spreading from a line source, with first-order loss, $(s, m) = (1, 3)$, yielding:

$$0 \leqslant \rho \leqslant \left[\frac{10}{3} U_0^3\right]^{1/2} = \rho_0,$$

$$U(\rho) = U_0\left[1-\left(\frac{\rho}{\rho_0}\right)^2\right]^{1/3}. \tag{37}$$

$$Q = \frac{\Gamma(4/3)}{\Gamma(11/6)}\left[\frac{10\pi}{3}\right]^{1/2} U_0^{5/2} \approx 3.074 U_0^{5/2}. \tag{38}$$

$$T(t) = \mathrm{e}^{-t}[(1-\mathrm{e}^{-3t})/3]^{-1/5} \tag{39}$$

$$R(t) = [(1-\mathrm{e}^{-3t})/3]^{1/5}. \tag{40}$$

4.4. *Gravity spreading without loss*

Axisymmetric spreading of a liquid lens on a horizontal solid surface, without loss, leads to the special form of (1) with $s = 2$, $m = 3$, and $k = 0$. Its solution [8] is the appropriate particular case of the well-known general instantaneous source solution [1–4]. It follows from the discussion of Section 4.1 that, with suitable definition of the symbols, the same solution applies also to spreading of the lens, without loss, on an immiscible liquid surface.

Similarly, one-dimensional gravity spreading without loss, from a line source on either type of surface, leads to the analogous formulation, and well-known solution, for $s = 1$, $m = 3$.

5. DIFFUSION WITH GAIN

Some thermal and biological problems involve diffusion with gain, not loss. The minus on the right of (1) and (3) is then replaced by a plus. The analysis goes similarly, but the appropriate substitutions are now

$$\theta(r, t) = u(r, \tau)\,\mathrm{e}^t, \tag{41}$$

$$\tau = m^{-1}(\mathrm{e}^{mt}-1), \quad 0 \leqslant \tau \leqslant \infty. \tag{42}$$

The solutions then become

$$T(t) = \mathrm{e}^t[m^{-1}(\mathrm{e}^{mt}-1)]^{-s/(sm+2)}; \tag{43}$$

$$R(t) = [m^{-1}(\mathrm{e}^{mt}-1)]^{1/(sm+2)}; \tag{44}$$

$$q(t) = Q\,\mathrm{e}^t. \tag{45}$$

Equation (45) is consistent with material being gained through a first-order process.

5.1. *Time-course of concentration scale*

The scale of concentration in the slug is proportional to T. T decreases initially, passes through a minimum, and ultimately increases exponentially. In the limit of small t

$$T(t) \approx t^{-s/(sm+2)}\left(1+\frac{(sm+4)t}{2(sm+2)}\right), \tag{46}$$

approximating classical behavior [1–4]. At large t

$$T(t) \approx m^{s/(sm+2)}\,\mathrm{e}^{2t/(sm+2)}\left(1+\frac{s}{sm+2}\mathrm{e}^{-mt}\right), \tag{47}$$

the power-law decrease giving way to exponential increase. We find that T attains its minimum value

$$T_{\min} = \left(\frac{s}{2}\right)^{-2/(sm+2)}\left(\frac{sm+2}{2}\right)^{1/m} \tag{48}$$

when

$$t = m^{-1}\ln\left(\frac{sm+2}{2}\right). \tag{49}$$

Figure 5 compares for $(s, m) = (2, 2)$, the $T(t)$ functions for loss, material conservation ($k = 0$), and gain.

5.2. *Slug length-scale*

The length-scale of the slug is proportional to R. R increases initially as a power of t and ultimately exponentially. At small t

$$R(t) \approx t^{1/(sm+2)}\left(1+\frac{mt}{2(sm+2)}\right), \tag{50}$$

also approximating the classical result [1–4]. At large t

$$R(t) \approx m^{-1/(sm+2)}\,\mathrm{e}^{mt/(sm+2)}\left(1-\frac{1}{sm+2}\mathrm{e}^{-mt}\right). \tag{51}$$

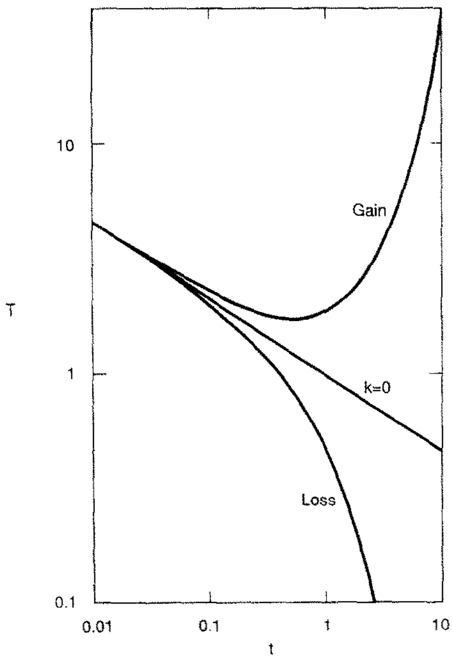

FIG. 5. The function $T(t)$ for $(s, m) = (2, 2)$. Comparison for loss, material conservation ($k = 0$), and gain. The concentration scale is proportional to T.

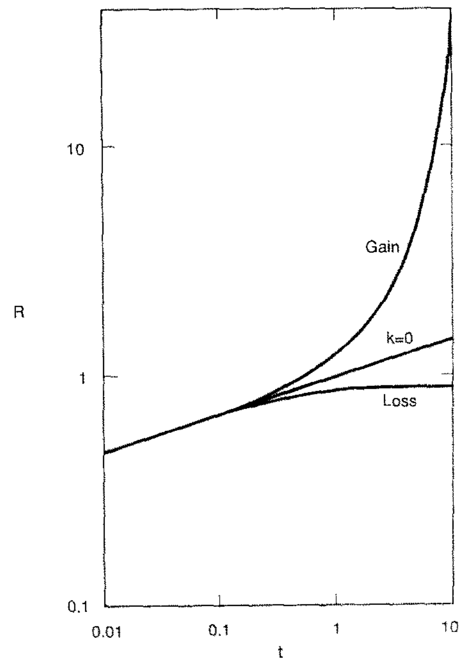

FIG. 6. The function $R(t)$ for $(s, m) = (2, 2)$. Comparison for loss, material conservation ($k = 0$), and gain. The slug length-scale is proportional to R.

Figure 6 makes for $R(t)$ the same comparison as Fig. 5 does for $T(t)$.

6. DISCUSSION

The foregoing exact solutions for nonlinear diffusion ($s > 0$, $m > 0$) with first-order loss from an instantaneous source all give slug radius exponentially approaching a finite maximum. Concentration approaches zero exponentially as $t \to \infty$ and the slug vanishes. The first-order loss rate decreases with θ fast enough to ensure persistence of the slug at all finite times. This contrasts with the case of loss rate proportional to θ^n ($0 \leqslant n < 1$), when slug radius increases to a maximum and then shrinks to zero in finite time [5, 11]. The physical explanation of the latter case is that loss rate at small θ remains large enough to ensure slug extinction in finite time.

A striking aspect of the solutions with gain (Section 5) is the persistence of the effects of the initial conditions. This contrasts with the disappearance of the effect of initial slug quantity as $t \to \infty$ from the solutions for gain $\propto \theta^n$ ($0 \leqslant n < 1$) [11]. Presumably the difference arises from the larger rate of gain at small θ in that case.

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REFERENCES

1. G. I. Barenblatt, On some unsteady motions of a liquid and a gas in a porous medium, *Akad. Nauk SSSR Prikl. Math. i Mekh.* **16,** 67–78 (1952).
2. R. E. Pattle, Diffusion from an instantaneous point source with a concentration-dependent diffusion coefficient, *Q. J. Mech. Appl. Math.* **12,** 407–409 (1959).
3. J. R. King, Exact similarity solutions to some nonlinear diffusion equations, *J. Phys. A. Math. Gen.* **23,** 3681–3697 (1990).
4. J. R. Philip, Fast diffusion with loss at infinity, *J. Australian Math. Soc., Ser. B* (in press).
5. C. A. Miller and C. J. van Duijn, Similarity solutions for gravity-dominated spreading of a lens of organic contaminant, *Environmental Studies, IMA Volumes in Mathematics and its Applications*. Springer, Heidelberg (in press).
6. H. C. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*. Clarendon, Oxford (1947).
7. J. R. Philip, An application of the diffusion equation to viscous motion with a free surface, *Australian J. Phys.* **9,** 570–573 (1956).
8. J. Lopez, C. A. Miller and E. Ruckenstein, Spreading kinetics of liquid drops on solids, *J. Colloid Interface Sci.* **56,** 460–467 (1976).
9. W. Badon-Ghyben, Nota in verband met de voorgenomen putboring nabij Amsterdam, *Tijdschrift Koninglijk Inst. Ing.* 21pp. (1888).
10. A. Herzberg, Die Wasserversorgung einiger Nordseebäder, *J. Gasbeleucht. Wasserversorg.* **44,** 815–819 and 842–844 (1901).
11. J. R. Philip, Exact solutions for nonlinear diffusion with nonlinear loss, *ZAMP* (in press).